

BRST PROPERTIES OF SPIN FIELDS***L. Dolan and S. Horvath**

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ABSTRACT

For the closed superstring, spin fields and bi-spinor states are defined directly in four spacetime dimensions, in van der Waerden notation (dotted and undotted indices). Explicit operator product expansions are given, including those for the internal superconformal field theory, which are consistent with locality and BRST invariance for the string vertices. The most general BRST picture changing for these fields is computed. A covariant notation for the spin decomposition of these states is developed in which non-vanishing polarizations are selected automatically. The kinematics of the three-gluon dual model amplitude in both the Neveu-Schwarz and Ramond sectors in the Lorentz gauges is calculated and contrasted. Modular invariance and enhanced gauge symmetry of four-dimensional models incorporating these states is described.

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1. Introduction

Conformal spin fields corresponding to physical states in the Ramond sector of the superstring are analyzed directly in four spacetime dimensions. BRST properties of spin fields and superconformal fields, associated with states in the Neveu-Schwarz sector, are compared. In sect. 2, we describe the conditions on the worldsheet supercurrent imposed by BRST invariance of the vertex operators and physical states. In sect. 3, we give a convenient notation for massless spinor states. This is useful in discussing a covariant spin decomposition of bi-spinor states and their relation to circularly polarized polarization vectors. These are then used to describe the spin decomposition of the general graviton tensor, and the vector-spinor states' decomposition into gravitinos and massless spin- $\frac{1}{2}$ states. Examples of the bi-spinor states are mesons from the Ramond-Ramond sector of the Type II superstring. In sect. 4, operator product expansions which maintain the locality of the massless fermion emission vertices are set up for spacetime and internal spin fields. For these conventional vertices, BRST picture changing, including the superconformal ghost number picture $-\frac{3}{2}$, is computed; and its dependence on the choice of supercurrent demonstrated. Tree amplitudes are calculated, and amplitudes involving Ramond states have the property that non-vanishing polarizations are selected automatically. In sect. 5, the kinematics of three-point amplitudes of massless particles are analyzed, leading to the identification of couplings for Neveu-Schwarz and Ramond mesons. In sect. 6, we present a set of tree amplitudes involving massless gauge bosons coming from the Ramond-Ramond sector, which exhibit a non-abelian structure. We suggest a possible mechanism involving a non-hermitian piece of the internal worldsheet supercurrent to include these states in a superstring model satisfying perturbative spacetime unitarity. In sect. 7 we discuss a modular invariant partition function. Conclusions are contained in sect. 8. There are two appendices concerned with the two-dimensional sigma matrices used in the Weyl/van der Waerden description of the spacetime gamma matrices and with the construction of higher-dimensional gamma matrices in the Weyl representation.

2. Ramond states

We consider the vertex operator directly in four dimensions

$$V_{-\frac{3}{2}}(k, z) = v^{\dot{\alpha}}(k) S_{\dot{\alpha}}(z) e^{ik \cdot X(z)} \mathcal{S}(z) e^{-\frac{3}{2}\phi(z)}. \quad (2.1)$$

A suitable choice of supercurrent is given by:

$$F(z) = a_{\mu}(z) h^{\mu}(z) + \bar{F}(z) \quad (2.2)$$

where $0 \leq \mu \leq 3$ and $\bar{F}(z)$ corresponds to internal degrees of freedom. The vertex operator for the Ramond states in the canonical $q = -\frac{1}{2}$ superconformal ghost picture is given by picture changing^[1,2] for $k \cdot \frac{1}{\sqrt{2}} \gamma v \sim u$ where $k \cdot \frac{1}{\sqrt{2}} \gamma u = 0$ by

$$\begin{aligned} V_{-\frac{1}{2}}(k, \zeta) &= \lim_{z \rightarrow \zeta} e^{\phi(z)} F(z) V_{-\frac{3}{2}}(k, \zeta) \\ &= [u^{\alpha}(k) S_{\alpha}(\zeta) e^{ik \cdot X(\zeta)} \mathcal{S}(\zeta) \\ &\quad + \lim_{z \rightarrow \zeta} (z - \zeta)^{\frac{3}{2}} \bar{F}(z) v^{\dot{\alpha}}(k) S_{\dot{\alpha}}(\zeta) e^{ik \cdot X(\zeta)} \mathcal{S}(\zeta)] e^{-\frac{1}{2}\phi(\zeta)}. \end{aligned} \quad (2.3)$$

BRST invariance of a vertex operator requires its commutator with the BRST charge Q to be a total divergence; for a vertex operator in the $q = -\frac{3}{2}$ superconformal ghost picture such as (2.1), this invariance is assured whenever its operator product with the supercurrent has at most a $(z - \zeta)^{-\frac{3}{2}}$ singularity. We see this as follows:

The ghost superfields^[1-3] are $B(z) = \beta(z) + \theta b(z)$ and $C(z) = c(z) + \theta \gamma(z)$ with conformal spin $h_\beta = \frac{3}{2}$, $h_c = -1$. Then $h_b = 2$ and $h_\gamma = -\frac{1}{2}$. The modings on the Ramond sector and the commutation relations are $\{b_n, c_m\} = \delta_{n,-m}$, $[\beta_n, \gamma_m] = \delta_{n,-m}$, for $n, m \in \mathbb{Z}$. Normal ordering is defined by putting the annihilation operators b_n for $n \geq -1$, c_n for $n \geq 2$ to the right of the creation operators b_n for $n \leq -2$, c_n for $n \leq 1$; then

$$b(z)c(\zeta) = \times b(z)c(\zeta) \times + \frac{1}{z - \zeta} \quad ; \quad c(z)b(\zeta) = \times c(z)b(\zeta) \times + \frac{1}{z - \zeta}. \quad (2.4)$$

This is a natural definition for normal ordering as the “vacuum” expectation value of this normal ordered product including its finite part is zero. For the superconformal ghosts, normal ordering is defined analogously so that

$$\beta(z)\gamma(\zeta) = \times \beta(z)\gamma(\zeta) \times - \frac{1}{z - \zeta} \quad ; \quad \gamma(z)\beta(\zeta) = \times \gamma(z)\beta(\zeta) \times + \frac{1}{z - \zeta}. \quad (2.5)$$

In order to make contact with the field $\phi(z)$ appearing in (2.1), we can then “bosonize” this system, i.e. write it as a theory where operators are associated with vectors q in the weight lattice of some algebra. For the bosonic β, γ conformal field theory, we define the boson fields

$$\phi(z)\phi(\zeta) =: \phi(z)\phi(\zeta) : - \ln(z - \zeta) \quad (2.6)$$

so that

$$: e^{\phi(z)} :: e^{\phi(\zeta)} : =: e^{\phi(z)} e^{\phi(\zeta)} : (z - \zeta)^{-1}. \quad (2.7)$$

Because $: e^{\phi(z)} :$ is a fermion field and $\beta(z), \gamma(z)$ are bosons, an additional bosonic field $\chi(z)$ is introduced and

$$\begin{aligned} \gamma(z) &=: e^{\phi(z)} :: e^{-\chi(z)} : \quad ; \quad J(z) = - \times \beta \gamma \times = \partial \phi \\ \beta(z) &=: e^{-\phi(z)} :: \partial e^{\chi(z)} : \end{aligned} \quad (2.8)$$

Here

$$\chi(z)\chi(\zeta) =: \chi(z)\chi(\zeta) : + \eta^{\mu\nu} \ln(z - \zeta). \quad (2.9)$$

Because the $\beta\gamma$ spectrum is unbounded from below, it is useful to define an infinite number of $\beta\gamma$ ‘vacua’ $|q\rangle_{\beta\gamma}$ where $\beta_n|q\rangle = 0, n > -q - \frac{3}{2}$, $\gamma_n|q\rangle = 0, n \geq q + \frac{3}{2}$, where $|q\rangle_{\beta\gamma} = e^{q\phi(0)}|0\rangle_{\beta\gamma}$ and $L_0^{\beta\gamma}|q\rangle = -\frac{1}{2}q(q+2)|q\rangle$. The bosonic β, γ ghost system has two sectors: one is Neveu-Schwarz where $q \in \mathbb{Z}$, and the fields $\beta(z)$, $\gamma(z)$, and $: e^{\phi(z)} :$ are periodic, i.e. half-integrally moded; the other sector is Ramond, where $q \in \mathbb{Z} + \frac{1}{2}$, and the fields $\beta(z)$, $\gamma(z)$, and $: e^{\phi(z)} :$ are anti-periodic, i.e. integrally moded. We note that the conformal fields $: e^{q\phi(z)} :$ for q odd have the same periodicity and statistics as the supercurrent. Their

conformal dimensions given by $L_0^{\beta\gamma}|q\rangle = -\frac{1}{2}q(q+2)|q\rangle$ are $\frac{1}{2}$ for $q = -1$; $-\frac{3}{2}$ for $q = 1, -3$; and $-\frac{15}{2}$ for $q = 3, -5$; etc.

The superVirasoro ghost representation has central charge $c = -15$:

$$L(z) = -2 \times b \partial c \times - \times (\partial b) c \times - \frac{3}{2} \times \beta \partial \gamma \times - \frac{1}{2} \times (\partial \beta) \gamma \times \quad (2.10a)$$

$$F(z) = b\gamma - 3\beta\partial c - 2(\partial\beta)c. \quad (2.10b)$$

For the $N = 1$ worldsheet supersymmetry system, the BRST charge $Q \equiv \frac{1}{2\pi i} \oint dz Q(z)$ is given from the general form

$$Q(z) \sim c(L^{\text{matter}} + \frac{1}{2}L^{\text{ghost}}) - \gamma \frac{1}{2}(F^{\text{matter}} + \frac{1}{2}F^{\text{ghost}}) \quad (2.11a)$$

by the BRST current

$$\begin{aligned} Q(z) &= Q_0(z) + Q_1(z) + Q_2(z) \\ Q_0(z) &= cL^{X,\psi} - \times cb\partial c \times + \frac{3}{2}\partial^2 c + cL^{\beta\gamma} + \partial(\frac{3}{4}\times\gamma\beta\times) \\ Q_1(z) &= -\gamma\frac{1}{2}F^{X,\psi} \\ Q_2(z) &= -\frac{1}{4}\gamma b\gamma. \end{aligned} \quad (2.11b)$$

Here the matter fields $L^{X,\psi}(z)$ and $F^{X,\psi}(z)$ close the superconformal algebra with $c = 15$; and L^{ghost} and F^{ghost} denoted in (2.11a) are given in (2.10). From the operator product expansion of the BRST current with itself it follows that

$$Q^2 = \frac{1}{2}\{Q, Q\} = 0. \quad (2.12)$$

This signals the conservation of the BRST charge and allows us to make different gauge choices. The commutator which vanishes to insure BRST invariance of the physical states corresponding to (2.1) is

$$[Q, V_{-\frac{3}{2}}(k, z)] = [Q_0, V_{-\frac{3}{2}}(k, z)] + [Q_1, V_{-\frac{3}{2}}(k, z)] + [Q_2, V_{-\frac{3}{2}}(k, z)]. \quad (2.13a)$$

By inspection, we find

$$[Q_0, V_{-\frac{3}{2}}(k, z)] = \frac{d}{dz}(c(z)V_{-\frac{3}{2}}(k, z)); \quad [Q_2, V_{-\frac{3}{2}}(k, z)] = 0 \quad (2.13b)$$

and

$$\begin{aligned} Q_1(z)V_{-\frac{3}{2}}(k, \zeta) &= -\frac{1}{2} : e^{-\chi(z)} : e^{\phi(z)} F(z) V_{-\frac{3}{2}}(k, \zeta) \\ &= -\frac{1}{2} : e^{-\chi(z)} : (z - \zeta)^{\frac{3}{2}} e^{-\frac{1}{2}\phi(z)} F(z) v^{\dot{\alpha}}(k) S_{\dot{\alpha}}(z) e^{ik \cdot X(z)} \mathcal{S}(z) \\ &= \text{regular terms} \end{aligned} \quad (2.13c)$$

so that

$$[Q_1, V_{\frac{3}{2}}(k, z)] = 0. \quad (2.13d)$$

3. Notation for the product of two spinors

In a Weyl representation, the four-dimensional γ matrix Clifford algebra given by $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ with $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ can be represented as

$$(\gamma^\mu)^A_B = \begin{pmatrix} 0 & (\bar{\sigma}^\mu)^{\alpha}_{\dot{\beta}} \\ (\sigma^\mu)^{\dot{\alpha}}_{\beta} & 0 \end{pmatrix}; \quad C^{AB} = \begin{pmatrix} (i\sigma^2)^{\alpha\beta} & 0 \\ 0 & (i\sigma^2)^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \quad (3.1)$$

where $\sigma^\mu = (\sigma^0, \sigma^i)$ and $\bar{\sigma}^\mu = (-\sigma^0, \sigma^i)$ are given by $\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the Pauli matrices σ^i . We define $\gamma^5 = (i\gamma^0\gamma^1\gamma^2\gamma^3)^A_B = \begin{pmatrix} (\sigma^0)^{\alpha}_{\dot{\beta}} & 0 \\ 0 & -(\sigma^0)^{\dot{\alpha}}_{\beta} \end{pmatrix}$. The charge conjugation matrices C^{AB} and $(C^{-1})_{AB}$ are tensors used to raise and lower indices: $C_{AD}^{-1}(\gamma^\mu)^D_B \equiv (\gamma^\mu)_{AB}$ and $C^{BD}(\gamma^\mu)^A_D \equiv (\gamma^\mu)^{AB}$. The transpose relation which defines C^{AB} is $C_{AB}^{-1}(\gamma^\mu)^B_C C^{CD} = -(\gamma^{\mu T})^D_A$ and it implies the matrices $(\gamma^\mu)^{AB}$ and $(\gamma^\mu)_{AB}$ are symmetric. For expressions involving the sigma matrices, see Appendix A.

Since we are in the Weyl representation, we can use van der Waerden notation^[4,5] for spinor indices $1 \leq \alpha, \dot{\alpha} \leq 2$. The two linearly independent solutions to the massless Dirac equation $k \cdot \gamma u^\ell(k) = 0$ are now given by solutions of the Weyl equations

$$k_\mu \sigma^{\mu\dot{\alpha}}_{\beta} u^{1\beta} = 0 \quad k_\mu \bar{\sigma}^{\mu\alpha}_{\dot{\beta}} u^{2\dot{\beta}} = 0 \quad (3.2a)$$

as

$$u^{1\beta}(k) = \begin{pmatrix} k^0 + k^3 \\ k^1 + ik^2 \end{pmatrix} (k^0 + k^3)^{-\frac{1}{2}}; \quad u^{2\dot{\beta}}(k) = \begin{pmatrix} -k^1 + ik^2 \\ k^0 + k^3 \end{pmatrix} (k^0 + k^3)^{-\frac{1}{2}}. \quad (3.2b)$$

We define two additional spinors $v^\ell(k)$ by $k \cdot \gamma v^\ell \sim u^\ell$ *i.e.*

$$\frac{1}{\sqrt{2}} k_\mu \bar{\sigma}^{\mu\alpha}_{\dot{\beta}} v^{1\dot{\beta}} = u^{1\alpha} \quad \frac{1}{\sqrt{2}} k_\mu \sigma^{\mu\dot{\alpha}}_{\beta} v^{2\beta} = -u^{2\dot{\alpha}} \quad (3.3a)$$

as

$$v^{1\dot{\beta}}(k) = \begin{pmatrix} k^0 + k^3 \\ k^1 + ik^2 \end{pmatrix} (2(k^0)^2(k^0 + k^3))^{-\frac{1}{2}}; \quad v^{2\beta}(k) = \begin{pmatrix} -k^1 + ik^2 \\ k^0 + k^3 \end{pmatrix} (2(k^0)^2(k^0 + k^3))^{-\frac{1}{2}}. \quad (3.3b)$$

Note that formally $v^{1\dot{\beta}} = \frac{1}{\sqrt{2k^0}} u^{1\beta}$ and $v^{2\beta} = \frac{1}{\sqrt{2k^0}} u^{2\dot{\beta}}$. In (3.2,3.3) we have $k_\mu k^\mu = 0$. The spin decomposition of the Weyl bispinors into the two helicity states of the massless vector is as follows:

$$\begin{aligned} u^{1\alpha} u^{1\beta} &= -\epsilon^+_{\lambda} k_{\kappa} (\bar{\sigma}^{\lambda} \sigma^{\kappa} \sigma^2)^{\alpha\beta} \\ u^{2\dot{\alpha}} u^{2\dot{\beta}} &= \epsilon^-_{\lambda} k_{\kappa} (\sigma^{\lambda} \bar{\sigma}^{\kappa} \sigma^2)^{\dot{\alpha}\dot{\beta}}. \end{aligned} \quad (3.4)$$

We also find that

$$\begin{aligned} u^{1\alpha} v^{1\dot{\beta}} &= \epsilon_{\lambda}^{+} (\bar{\sigma}^{\lambda} \sigma^2)^{\alpha\dot{\beta}} \sqrt{2}; & u^{2\dot{\delta}} v^{2\gamma} &= \epsilon_{\lambda}^{-} (\sigma^{\lambda} \sigma^2)^{\dot{\delta}\gamma} \sqrt{2} \\ v^{1\dot{\alpha}} u^{1\beta} &= \epsilon_{\lambda}^{+} (\sigma^{\lambda} \sigma^2)^{\dot{\alpha}\beta} \sqrt{2}; & v^{2\delta} u^{2\dot{\gamma}} &= \epsilon_{\lambda}^{-} (\bar{\sigma}^{\lambda} \sigma^2)^{\delta\dot{\gamma}} \sqrt{2} \end{aligned} \quad (3.5)$$

where the expressions in (3.5) are defined only up to a gauge transformation $\epsilon_{\lambda} \rightarrow \epsilon_{\lambda} + k_{\lambda}$. In the Lorentz gauges defined by $k \cdot \epsilon^{\pm}(k) = 0$, we have that

$$i\epsilon^{\mu\nu\lambda\rho} \epsilon_{\lambda}^{\pm} k_{\rho} = \pm(\epsilon^{\mu\pm} k^{\nu} - \epsilon^{\nu\pm} k^{\mu}) \quad (3.6)$$

holds generally for the circularly polarized polarization vectors^[6]. The expressions (3.4,3.5) satisfy (3.2a),(3.3a) with use of (3.6). We note that (3.4) describes only two polarizations since we can show that

$$\epsilon_{\lambda}^{-} k_{\kappa} (\bar{\sigma}^{\lambda} \sigma^{\kappa} \sigma^2)^{\alpha\beta} = 0; \quad \epsilon_{\lambda}^{+} k_{\kappa} (\sigma^{\lambda} \bar{\sigma}^{\kappa} \sigma^2)^{\dot{\delta}\gamma} = 0. \quad (3.7)$$

The proof of (3.7) is as follows:

Consider

$$\sigma_{\mu} \epsilon_{\lambda}^{-} k_{\kappa} (\bar{\sigma}^{\lambda} \sigma^{\kappa} \sigma^2)^{\alpha\beta} = (\epsilon_{\mu}^{-} k_{\sigma} - \epsilon_{\sigma}^{-} k_{\mu} + i\epsilon_{\lambda}^{-} k_{\kappa} \epsilon^{\lambda\kappa}_{\mu\sigma}) (\sigma^{\sigma} \sigma^2)^{\alpha\beta} = 0 \quad (3.8)$$

which for $\mu = 0$ is the left equation in (3.7).

The spin decomposition of the Weyl bispinors into the two spin zero states is

$$\begin{aligned} u^{1\alpha} u^{2\dot{\beta}} &= k_{\kappa} (i\bar{\sigma}^{\kappa} \sigma^2)^{\alpha\dot{\beta}} \\ u^{2\dot{\delta}} u^{1\gamma} &= k_{\kappa} (i\sigma^{\kappa} \sigma^2)^{\dot{\delta}\gamma}. \end{aligned} \quad (3.9)$$

We also note here the spin decomposition of the spin-vector state $\psi_{\mu}^A = \epsilon_{\mu}^{+} u^A$ separated into its spin- $\frac{3}{2}$ and spin- $\frac{1}{2}$ content. This is simple in van der Waerden notation and eliminates the need for introducing^[7,8] a noncovariant momentum vector \bar{k} where $k \cdot \bar{k} = 1$. We find the spin- $\frac{3}{2}$ part to be given by

$$\psi_{\mu}^{+\alpha} = \epsilon_{\mu}^{+} u^{1\alpha} \quad \text{helicity} = \frac{3}{2}; \quad \psi_{\mu}^{-\dot{\beta}} = \epsilon_{\mu}^{-} u^{2\dot{\beta}} \quad \text{helicity} = -\frac{3}{2} \quad (3.10)$$

since the spin-vectors in (3.10) satisfy the on-shell $k^2 = 0$ Rarita-Schwinger equation

$$k \cdot \psi^{\pm} = 0; \quad k \cdot \gamma \psi_{\mu}^{\pm} = 0; \quad \gamma \cdot \psi = 0. \quad (3.11)$$

For example, for the spin- $\frac{3}{2}$ helicity,

$$\gamma \cdot \psi^{\pm} = \sigma^{\mu\dot{\alpha}}_{\alpha} \epsilon_{\mu}^{+} u^{1\alpha} = 0. \quad (3.12)$$

To prove (3.12), consider

$$\begin{aligned}\sigma^{\mu\dot{\alpha}}_{\alpha}\epsilon_{\mu}^{+}u^{1\alpha}v^{1\dot{\beta}} &= \epsilon_{\mu}^{+}\sigma^{\mu\dot{\alpha}}_{\alpha}\epsilon_{\lambda}^{+}(\bar{\sigma}^{\lambda}\sigma^2)^{\dot{\alpha}\dot{\beta}}\sqrt{2} \\ &= \sqrt{2}\epsilon^{+}\cdot\epsilon^{+}\sigma^2 = 0.\end{aligned}\tag{3.13}$$

The spin- $\frac{1}{2}$ part is

$$\chi_{\mu}^{+\dot{\beta}} = \epsilon_{\mu}^{+}u^{2\dot{\beta}} \quad \text{helicity} = \frac{1}{2}; \quad \chi_{\mu}^{-\alpha} = \epsilon_{\mu}^{-}u^{1\alpha} \quad \text{helicity} = -\frac{1}{2}\tag{3.14}$$

since for the spin vectors in (3.14) we have

$$k\cdot\chi = 0; \quad k\cdot\gamma\chi_{\mu} = 0; \quad \gamma\cdot\chi \neq 0.\tag{3.15}$$

The connection between the Lorentz covariant notation developed here and the k, \bar{k} notation used in [7,8] is seen explicitly in (3.16a,b). As an example, consider the helicity equal to $+\frac{1}{2}$ part, $\chi_{\mu}^{+\dot{\beta}} = \epsilon_{\mu}^{+}u^{2\dot{\beta}}$. Then

$$\chi_{\mu}^{+\dot{\beta}} = \epsilon_{\mu}^{+}u^{2\dot{\beta}}\tag{3.16a}$$

$$= -\frac{i}{2}(\sigma_{\mu} - k_{\mu}\bar{k}\cdot\sigma)^{\dot{\beta}}_{\beta}u^{1\beta}\tag{3.16b}$$

so that

$$\begin{aligned}\gamma\cdot\chi^{+} &= \epsilon_{\mu}^{+}\bar{\sigma}^{\mu\alpha}_{\dot{\beta}}u^{2\dot{\beta}} = -iu^{1\alpha} \\ &= -\frac{i}{2}(\bar{\sigma}^{\mu}(\sigma_{\mu} - k_{\mu}\bar{k}\cdot\sigma))^{\dot{\beta}}_{\beta}u^{1\beta}.\end{aligned}\tag{3.17}$$

The general graviton tensor $\epsilon_{\mu\nu}$ in this notation is given by:

$$\begin{aligned}h_{\mu\nu} &= \frac{1}{2}(\epsilon_{\mu\nu} + \epsilon_{\nu\mu}) - \frac{1}{d-2}\epsilon^{\rho}_{\rho}(\eta_{\mu\nu} - k_{\mu}\bar{k}_{\nu} - \bar{k}_{\mu}k_{\nu}) = \epsilon_{\mu}^{+}\epsilon_{\nu}^{+} \text{ or } \epsilon_{\mu}^{-}\epsilon_{\nu}^{-} \\ B_{\mu\nu} &= \frac{1}{2}(\epsilon_{\mu\nu} - \epsilon_{\nu\mu}) = \frac{1}{2}(\epsilon_{\mu}^{+}\epsilon_{\nu}^{-} - \epsilon_{\mu}^{-}\epsilon_{\nu}^{+}) \\ D_{\mu\nu} &= \frac{1}{\sqrt{d-2}}(\eta_{\mu\nu} - k_{\mu}\bar{k}_{\nu} - \bar{k}_{\mu}k_{\nu}) = \frac{2}{\sqrt{d-2}}(\epsilon_{\mu}^{+}\epsilon_{\nu}^{-} + \epsilon_{\mu}^{-}\epsilon_{\nu}^{+}).\end{aligned}\tag{3.18}$$

Here $\epsilon_{\mu\nu}(k)$ is polarization tensor which satisfies $k^{\mu}\epsilon_{\mu\nu}(k) = k^{\nu}\epsilon_{\mu\nu}(k) = 0$, and represents either a (symmetric traceless) graviton $h_{\mu\nu}$, an anti-symmetric tensor $B_{\mu\nu}$ which is a pseudo-scalar in four dimensions, or a (transverse diagonal) scalar dilaton D . We remark that the right-hand terms in (3.18) provide a covariant spin decomposition of the spin-2 tensor since they are independent of \bar{k} . They follow from the Lorentz gauge condition $k^{\mu}\epsilon_{\mu}^{\pm} = 0$ and the normalization conditions $\epsilon_{\mu}^{+}\epsilon^{\mu+} = \epsilon_{\mu}^{-}\epsilon^{\mu-} = 0$, $\epsilon_{\mu}^{+}\epsilon^{\mu-} = 1$.

4. BRST invariance and picture changing

In four spacetime dimensions, the vertex operators associated with massless Ramond states can be described by

$$\begin{aligned} V_{-\frac{3}{2}}^{(1)}(k, z) &= v^{1\dot{\alpha}}(k) S_{\dot{\alpha}}(z) e^{ik \cdot X(z)} f^{\ell} \Sigma_{\ell}(z) e^{-\frac{3}{2} \phi(z)} \\ V_{-\frac{3}{2}}^{(2)}(k, z) &= v^{2\alpha}(k) S_{\alpha}(z) e^{ik \cdot X(z)} f^{\dot{\ell}} \Sigma_{\dot{\ell}}(z) e^{-\frac{3}{2} \phi(z)} \end{aligned} \quad (4.1)$$

where $v^{1\dot{\alpha}}, v^{2\alpha}$ are given in (3.3b). In order to maintain locality for the string vertices, we choose the following operator product expansions:

$$\begin{aligned} S_{\alpha}(z) S_{\beta}(\zeta) &= (z - \zeta)^{-\frac{1}{2}} C_{\alpha\beta}^{-1} + \dots \\ S_{\alpha}(z) S_{\dot{\beta}}(\zeta) &= (z - \zeta)^0 \gamma_{\alpha\dot{\beta}}^{\mu} \frac{1}{\sqrt{2}} \psi_{\mu}(\zeta) + \dots \\ S_{\dot{\alpha}}(z) S_{\beta}(\zeta) &= (z - \zeta)^0 \gamma_{\dot{\alpha}\beta}^{\mu} \frac{1}{\sqrt{2}} \psi_{\mu}(\zeta) + \dots \\ S_{\dot{\alpha}}(z) S_{\dot{\beta}}(\zeta) &= (z - \zeta)^{-\frac{1}{2}} C_{\dot{\alpha}\dot{\beta}}^{-1} + \dots \end{aligned} \quad (4.2a)$$

$$\begin{aligned} \Sigma_{\ell}(z) \Sigma_{\dot{n}}(\zeta) &= (z - \zeta)^{-\frac{3}{4}} C_{\ell\dot{n}}^{-1} + \dots \\ \Sigma_{\dot{\ell}}(z) \Sigma_n(\zeta) &= (z - \zeta)^{-\frac{3}{4}} C_{\dot{\ell}n}^{-1} + \dots \\ \Sigma_{\ell}(z) \Sigma_n(\zeta) &= (z - \zeta)^{-\frac{1}{4}} \Gamma_{\ell n}^a \frac{1}{\sqrt{2}} \psi_a(\zeta) + \dots \\ \Sigma_{\dot{\ell}}(z) \Sigma_{\dot{n}}(\zeta) &= (z - \zeta)^{-\frac{1}{4}} \Gamma_{\dot{\ell}\dot{n}}^a \frac{1}{\sqrt{2}} \psi_a(\zeta) + \dots \end{aligned} \quad (4.2b)$$

where the subleading terms are less singular by integer powers of $(z - \zeta)$, and the field $\psi^a(z)$ is understood to represent $\tilde{\gamma}^5 \otimes \psi^a(z)$, in order to maintain anticommutativity of $\psi^a(z)$ and $\psi^{\mu}(z)$. The operator relations above are not single-valued, but when taken in appropriate combinations with each other and the ghost fields, the resulting string vertices are local (in the sense of meromorphic operator product expansions), at least at zero momentum; and the momentum-dependent fields are local when the complete closed string holomorphic and antiholomorphic expressions are considered.

In (4.2b), we have chosen a free fermion form for part of the internal conformal field theory with $c = 3$. Here $1 \leq a \leq 6$ and we use a Weyl representation for the internal gamma matrices given for $1 \leq a \leq 3, 1 \leq \ell, \dot{\ell} \leq 4$ by

$$\Gamma^a = \begin{pmatrix} 0 & (\alpha^a)^{\ell}_{\dot{m}} \\ -(\alpha^a)^{\dot{\ell}}_m & 0 \end{pmatrix}; \Gamma^{a+3} = i \begin{pmatrix} 0 & (\beta^a)^{\ell}_{\dot{m}} \\ (\beta^a)^{\dot{\ell}}_m & 0 \end{pmatrix}; C = \begin{pmatrix} 0 & (I_4)^{\ell\dot{m}} \\ (I_4)^{\dot{\ell}m} & 0 \end{pmatrix}. \quad (4.3)$$

Also $\tilde{\gamma}^5 \equiv \gamma^5 (-1)^{\sum_{n>0} \psi_{-n}^{\mu} \psi_n^{\mu}}$ and $\tilde{\Gamma}^7 \equiv (i\Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6) (-1)^{\sum_{n>0} \psi_{-n}^a \psi_n^a}$ where $i\Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 \Gamma^5 \Gamma^6 = \begin{pmatrix} (I_4)^{\ell}_m & 0 \\ 0 & -(I_4)^{\dot{\ell}}_{\dot{m}} \end{pmatrix}$. The matrices α^a, β^a are real antisymmetric. See

Appendix B. In the Weyl representation, the charge conjugation and gamma matrices have the following properties: for four dimensions,

$$C_{\alpha\beta} = -C_{\beta\alpha}, \quad \gamma_{\alpha\dot{\beta}}^{\mu} = \gamma_{\dot{\beta}\alpha}^{\mu}, \quad \text{etc.} \quad (4.4a)$$

for six dimensions,

$$C_{\ell\dot{n}} = C_{\dot{n}\ell}, \quad \Gamma_{\ell n}^a = -\Gamma_{n\ell}^a, \quad \text{etc.} \quad (4.4b)$$

for ten dimensions,

$$C_{A\dot{B}} = -C_{\dot{B}A}, \quad \Gamma_{AB}^M = \Gamma_{BA}^M, \quad \text{etc.} \quad (4.4c)$$

For a general internal field theory, the expression $\Gamma_{\ell n}^a \frac{1}{\sqrt{2}} \psi_a(z)$ in (4.2b) is replaced by $\psi_{\ell n}(z)$, which has conformal weight $\frac{1}{2}$ and is antisymmetric in ℓ, n . To check the locality of the vertex operators in (4.1) at zero momentum, we use (4.2,4) to show

$$\begin{aligned} V_{-\frac{3}{2}}^{(1)}(z) V_{-\frac{3}{2}}^{(1)}(\zeta) &= (z - \zeta)^{-3} v^{1\dot{\alpha}}(k_1) v^{1\dot{\beta}}(k_2) C_{\dot{\alpha}\dot{\beta}}^{-1} f^{\ell} f^{\dot{n}} \Gamma_{\ell n}^a \frac{1}{\sqrt{2}} \psi_a(\zeta) e^{-3\phi(\zeta)} + \dots \\ &\cong -V_{-\frac{3}{2}}^{(1)}(\zeta) V_{-\frac{3}{2}}^{(1)}(z) \end{aligned} \quad (4.5a)$$

where in (4.5a) the operator product $V_{-\frac{3}{2}}^{(1)}(z) V_{-\frac{3}{2}}^{(1)}(\zeta)$ is defined for $|z| > |\zeta|$, the product $V_{-\frac{3}{2}}^{(1)}(\zeta) V_{-\frac{3}{2}}^{(1)}(z)$ is defined for $|\zeta| > |z|$ and \cong denotes equal in the sense of analytic continuation^[1,9]. Similary, we find local fermionic fields

$$V_{-\frac{3}{2}}^{(i)}(z) V_{-\frac{3}{2}}^{(j)}(\zeta) \cong -V_{-\frac{3}{2}}^{(j)}(\zeta) V_{-\frac{3}{2}}^{(i)}(z) \quad (4.5b)$$

for all $1 \leq i, j \leq 2$, using (4.1,4.2).

From (2.3), we see that by selecting a particular supercurrent, we can define $q = -\frac{1}{2}$ ghost picture fermion vertex operators. From (2.13c), we know that any supercurrent whose operator product with a fermion vertex operator has at most a $(z - \zeta)^{-\frac{3}{2}}$ singularity will ensure BRST invariance for the vertex operator. In this section, we choose a supercurrent (2.2) such that the internal $\bar{F}(z)$ has at most a $(z - \zeta)^{-\frac{1}{2}}$ singularity with (4.1). Then from (3.3a),

$$\begin{aligned} V_{-\frac{1}{2}}^{(1)}(k, \zeta) &= \lim_{z \rightarrow \zeta} e^{\phi(z)} F(z) V_{-\frac{3}{2}}^{(1)}(k, \zeta) \\ &= u^{1\alpha}(k) S_{\alpha}(\zeta) e^{ik \cdot X(\zeta)} f^{\ell} \Sigma_{\ell}(\zeta) e^{-\frac{1}{2}\phi(\zeta)}. \end{aligned} \quad (4.6a)$$

$$\begin{aligned} V_{-\frac{1}{2}}^{(2)}(k, \zeta) &= \lim_{z \rightarrow \zeta} e^{\phi(z)} F(z) V_{-\frac{3}{2}}^{(2)}(k, \zeta) \\ &= -u^{2\dot{\alpha}}(k) S_{\dot{\alpha}}(\zeta) e^{ik \cdot X(\zeta)} f^{\dot{\ell}} \Sigma_{\dot{\ell}}(\zeta) e^{-\frac{1}{2}\phi(\zeta)}. \end{aligned} \quad (4.6b)$$

BRST invariance of the vertex operators in the $q = -\frac{1}{2}$ picture (4.6) is assured from BRST invariance of the picture changed $q = -\frac{3}{2}$ operators and from (2.12), or can be checked directly. Locality holds from (4.2) for these operators as well:

$$\begin{aligned} V_{-\frac{1}{2}}^{(1)}(z) V_{-\frac{1}{2}}^{(1)}(\zeta) &= (z - \zeta)^{-1} u^{1\alpha}(k_1) u^{1\beta}(k_2) C_{\alpha\beta}^{-1} f^\ell f^n \Gamma_{\ell n}^a \psi_a(\zeta) e^{-\phi(\zeta)} + \dots \\ &\cong -V_{-\frac{1}{2}}^{(1)}(\zeta) V_{-\frac{1}{2}}^{(1)}(z) \end{aligned} \quad (4.7a)$$

where we note that unlike the Neveu-Schwarz case, the statistics of the fields associated with Ramond states does not change from one picture to another. In general, $V_{-\frac{1}{2}}^{(i)}(z) V_{-\frac{1}{2}}^{(j)}(\zeta) \cong -V_{-\frac{1}{2}}^{(j)}(\zeta) V_{-\frac{1}{2}}^{(i)}(z)$. Also, locality holds between fields in different pictures:

$$\begin{aligned} V_{-\frac{1}{2}}^{(1)}(z) V_{-\frac{3}{2}}^{(1)}(\zeta) &= (z - \zeta)^{-1} u^{1\alpha}(k_1) v^{1\beta}(k_2) \gamma_{\alpha\dot{\beta}}^\mu \psi_\mu(\zeta) f^\ell f^n \Gamma_{\ell n}^a \psi_a(\zeta)^{\frac{1}{2}} e^{-\phi(\zeta)} + \dots \\ &\cong -V_{-\frac{1}{2}}^{(1)}(\zeta) V_{-\frac{1}{2}}^{(1)}(z) \end{aligned} \quad (4.7b)$$

To establish (4.7b), we note as previously mentioned below (4.2) that the field $\psi^a(z)$ is understood to represent $\tilde{\gamma}^5 \otimes \psi^a(z)$, and we use $\gamma^5 u^1 \sim u^1$, $\gamma^5 v^1 \sim -v^1$, etc. In general, $V_{-\frac{1}{2}}^{(i)}(z) V_{-\frac{3}{2}}^{(j)}(\zeta) \cong -V_{-\frac{3}{2}}^{(j)}(\zeta) V_{-\frac{1}{2}}^{(i)}(z)$.

The vertex operators for the massless Neveu-Schwarz states in the canonical $q = -1$ superconformal ghost picture are

$$V_{-1}(k, z, \epsilon) = \epsilon \cdot \psi(z) e^{ik \cdot X(z)} e^{-\phi(z)} \quad (4.8a)$$

$$V_{-1}^a(k, z) = \tilde{\gamma}^5 \otimes \psi^a(z) e^{ik \cdot X(z)} e^{-\phi(z)} \quad (4.8b)$$

States in the Neveu-Schwarz matter system (i.e. without ghosts) form^[1] superconformal fields $V(z, \theta) = V_q(z) + \theta V_{q+1}(z)$ with upper and lower components related by

$$\begin{aligned} G(z) V_q(\zeta) &= (z - \zeta)^{-1} V_{q+1}(\zeta) \\ G(z) V_{q+1}(\zeta) &= (z - \zeta)^{-2} 2h_q V_q(\zeta) + (z - \zeta)^{-1} \partial V_q(\zeta). \end{aligned} \quad (4.9)$$

BRST invariance holds for both the vertices (4.8) since

$$[Q_0, V_{-1}(k, z)] = \frac{d}{dz} [c(z) V_{-1}(k, z)]; \quad [Q_2, V_{-1}(k, z)] = 0 \quad (4.10)$$

and

$$\begin{aligned} Q_1(z) V_{-1}(k, \zeta) &= -\frac{1}{2} : e^{-\chi(z)} : e^{\phi(z)} G(z) V_{-1}(k, \zeta) \\ &= -\frac{1}{2} : e^{-\chi(z)} : (z - \zeta)^1 G(z) V_{-1}^{\text{matter}}(k, \zeta) \\ &= -\frac{1}{2} : e^{-\chi(z)} : V_0^{\text{matter}}(k, \zeta) \\ &= \text{regular terms} \end{aligned} \quad (4.11a)$$

so that

$$[Q_1, V_{-1}k, z) = 0. \quad (4.11b)$$

In the $q = 0$ superconformal ghost picture, (4.8) is

$$\begin{aligned} V_0(k, z, \epsilon) &= \lim_{z \rightarrow \zeta} e^{\phi(z)} G(z) V_{-1}(k, \zeta \epsilon) \\ &= [k \cdot \psi(\zeta) \epsilon \cdot \psi(\zeta) + \epsilon \cdot a(\zeta)] e^{ik \cdot X(\zeta)} \end{aligned} \quad (4.12a)$$

$$\begin{aligned} V_0^a(k, z) &= \lim_{z \rightarrow \zeta} e^{\phi(z)} G(z) V_{-1}^a(k, \zeta) \\ &= [k \cdot \psi(\zeta) \tilde{\gamma}^5 \otimes \psi^a(\zeta) + (\lim_{z \rightarrow \zeta} (z - \zeta) \bar{F}(z) \tilde{\gamma}^5 \otimes \psi^a(\zeta))] e^{ik \cdot X(\zeta)}. \end{aligned} \quad (4.12b)$$

The tree correlation functions of BRST invariant vertex operators are independent of the distribution of ghost charges given that $\sum_i q_i = -2$. As an example we consider the coupling of the *general graviton tensor with two massless vector mesons*. For Neveu-Schwarz mesons we have

$$\begin{aligned} &\langle 0 | V_{-1}^a(k_1, z_1) V_0(k_2, z_2, \epsilon_2) V_{-1}^b(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\ &\cdot \langle 0 | V_{-1}(k_1, \bar{z}_1, \epsilon_1) V_0(k_2, \bar{z}_2, \epsilon_2) V_{-1}(k_3, \bar{z}_3, \epsilon_3) c(\bar{z}_1) c(\bar{z}_2) c(\bar{z}_3) | 0 \rangle \\ &= \delta^{ab} \epsilon_2^{\mu\nu} k_{3\mu} [k_{3\nu} \epsilon_1 \cdot \epsilon_3 + \epsilon_{3\nu} \epsilon_1 \cdot k_2 + \epsilon_{1\nu} \epsilon_3 \cdot k_1] \end{aligned} \quad (4.13)$$

where conformal ghost contributions^[1] are included, such as

$$\langle 0 | c(z_1) c(z_2) c(z_3) | 0 \rangle = (z_1 - z_2)(z_2 - z_3)(z_1 - z_3). \quad (4.14)$$

For Ramond-Ramond mesons we have

$$\begin{aligned} &\langle 0 | V_{-\frac{1}{2}}^{(2)}(k_1, z_1) V_{-1}(k_2, z_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\ &\cdot \langle 0 | V_{-\frac{1}{2}}^{(2)}(k_1, \bar{z}_1) V_{-1}(k_2, \bar{z}_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, \bar{z}_3) c(\bar{z}_1) c(\bar{z}_2) c(\bar{z}_3) | 0 \rangle \\ &= u^{2\dot{\delta}}(k_1) u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu} (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) u^{1\beta}(k_3) \epsilon_{2\nu} (-i\sigma^2 \bar{\sigma}^\nu)_{\beta\dot{\delta}} \frac{1}{2} f^{\dot{k}} f^{\dot{n}} C_{\dot{n}\ell}^{-1} f^\ell f^m C_{m\dot{k}}^{-1} \\ &= \frac{1}{8} \delta^{IJ} \epsilon_{1\rho}^- k_{1\sigma} \epsilon_{2\mu\nu} \epsilon_{3\lambda}^+ k_{3\kappa} \text{tr}(\sigma^\rho \bar{\sigma}^\sigma \sigma^\mu \bar{\sigma}^\lambda \sigma^\kappa \bar{\sigma}^\nu) \\ &= -\delta^{IJ} \epsilon_2^{\mu\nu} k_{3\mu} [k_{3\nu} \epsilon_1^- \cdot \epsilon_3^+ + \epsilon_{3\nu}^+ \epsilon_1^- \cdot k_2 + \epsilon_{1\nu}^- \epsilon_3^+ \cdot k_1] \end{aligned} \quad (4.15)$$

which up to a sign is the same as (4.13) for these polarizations. To evaluate (4.15), we have used a trace formula in Appendix A and various on-shell identities from sect. 5. In (4.15) only $\epsilon_{\mu\nu} = h_{\mu\nu}$ survives. We normalize the vertex operators by expanding $f_L^\ell f_R^m = \frac{1}{4} M_{\ell m}^J$ where $1 \leq J \leq 16$ and $M_{\ell m}^J$ are a complete set of sixteen linearly independent real four by four matrices $M^J = \{\alpha^a, \beta^a, \alpha^a \beta^b, I_4\}$ and $1(M^J)^2 = \pm I_4$, $\text{tr} M^J = 0$, $\text{tr} M^{I\dagger} M^J = 4\delta^{IJ}$. See Appendix B. In general, the evaluation of amplitudes in the BRST formalism is carried out by the calculation of standard correlation functions, whose singularity structure is set from the operator product expansions^[10,11]. For example, from (4.2) we find

$$\begin{aligned} \langle 0 | S_{\dot{\gamma}}(z_1) \psi^\mu(z_2) S_\alpha(z_3) | 0 \rangle &= \gamma_{\dot{\gamma}\alpha}^\mu \frac{1}{\sqrt{2}} (z_1 - z_2)^{-\frac{1}{2}} (z_2 - z_3)^{-\frac{1}{2}} \\ \langle 0 | \Sigma_{\dot{n}}(z_1) \Sigma_\ell(z_3) | 0 \rangle &= (z_1 - z_3)^{-\frac{3}{4}} C_{\dot{n}\ell}^{-1}. \end{aligned} \quad (4.16)$$

5. Kinematics of the three-point amplitude

In order to identify on-shell dual model amplitudes with conventional with conventional field theory couplings, we routinely use k_i^2 for massless particles, which together with momentum conservation in the three-point amplitudes, $k_1^\mu + k_2^\mu + k_3^\mu = 0$, implies

$$k_i \cdot k_j = 0. \quad (5.1)$$

In general, the dual model picks up a particular gauge^[7,8] to describe the graviton or gauge boson. For gravitons, the harmonic condition satisfied in any gauge

$$k^\mu \epsilon_{\mu\nu} = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu \quad (5.2)$$

arises in the dual amplitude in the harmonic gauge:

$$\epsilon^\mu{}_\mu = 0 \quad (5.3a)$$

so that

$$k^\mu \epsilon_{\mu\nu} = 0. \quad (5.3b)$$

Similarly, for gauge bosons, the string amplitudes pick up the Lorentz gauges described by (3.6), so that we use also

$$k^\mu \epsilon_\mu^\pm = 0. \quad (5.4)$$

In fact, further identities occur that are particularly useful in identifying amplitudes involving states in the Ramond-Ramond sector as standard field theory couplings^[6]. For the kinematics of the three-gluon amplitude we have

$$\epsilon_1^+ \cdot \epsilon_2^+ \epsilon_3^+ \cdot k_1 + \epsilon_2^+ \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2 + \epsilon_3^+ \cdot \epsilon_1^+ \epsilon_2^+ \cdot k_3 = 0 \quad (5.5a)$$

$$\epsilon_1^- \cdot \epsilon_2^- \epsilon_3^- \cdot k_1 + \epsilon_2^- \cdot \epsilon_3^- \epsilon_1^- \cdot k_2 + \epsilon_3^- \cdot \epsilon_1^- \epsilon_2^- \cdot k_3 = 0 \quad (5.5b)$$

when all the polarizations are the same, even for complex momenta. (For real momenta, the three-gluon amplitude vanishes for any choice of polarizations^[6,12], so one should be careful not to require real momenta, which is no problem due to analyticity.) For two polarizations of one helicity, and the third polarization with opposite helicity, then

$$\epsilon_1^+ \cdot \epsilon_3^+ \epsilon_2^- \cdot k_2 = 0 \quad (5.5c)$$

and

$$\epsilon_2^- \cdot k_1 \epsilon_1^+ \cdot k_2 = 0 \quad (5.5d)$$

so that

$$\epsilon_1^+ \cdot \epsilon_2^- \epsilon_3^+ \cdot k_1 + \epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2 + \epsilon_3^+ \cdot \epsilon_1^+ \epsilon_2^- \cdot k_3 = \epsilon_1^+ \cdot \epsilon_2^- \epsilon_3^+ \cdot k_1 + \epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2 \quad (5.5e)$$

with similar formulas for the other helicities. The proof of (5.5) is as follows: Consider

$$\begin{aligned} & i\epsilon^{\rho\kappa\mu\lambda}\epsilon_{1\rho}^+k_{2\kappa}\epsilon_{2\mu}^\pm\epsilon_{3\lambda}^+ \\ & = \pm(\epsilon_2^\pm\cdot\epsilon_3^+\epsilon_1^+\cdot k_2 + \epsilon_1^+\cdot\epsilon_2^\pm\epsilon_3^+\cdot k_1) \end{aligned} \quad (5.6a)$$

$$\begin{aligned} & = -i\epsilon^{\rho\kappa\mu\lambda}\epsilon_{1\rho}^+(k_{1\kappa} + k_{3\kappa})\epsilon_{2\mu}^\pm\epsilon_{3\lambda}^+ \\ & = -(\epsilon_3^+\cdot\epsilon_1^+\epsilon_2^\pm\cdot k_3 + \epsilon_1^+\cdot\epsilon_2^\pm\epsilon_3^+\cdot k_1) - (\epsilon_2^\pm\cdot\epsilon_3^+\epsilon_1^+\cdot k_2 + \epsilon_3^+\cdot\epsilon_1^+\epsilon_2^\pm\cdot k_3) \end{aligned} \quad (5.6b)$$

where (5.6a,b) are derived using (3.6). Equating (5.6a) with (5.6b) we find (5.5a,c). Replacing ϵ_3 with k_3 in (5.6), we also find

$$\epsilon_2^-\cdot k_3\epsilon_1^+\cdot k_3 = 0 \quad (5.7)$$

which is (5.6d) using (5.4). Equations (5.5) are invariant under gauge transformations $\epsilon_\mu \rightarrow k_\mu$ using (5.1).

We note that amplitudes involving states in the Ramond sector are automatically zero for certain combinations of polarizations. This is contrasted with Neveu-Schwarz amplitudes which can be calculated initially independent of the choice of polarizations. A subsequent closer look at the Lorentz gauge condition (3.6) then reduces the kinematics of the Neveu-Schwarz amplitudes to those of the Ramond amplitudes for identical couplings. (See however the remark below (5.14)). From (4.14), we have the coupling of the general graviton tensor with two massless Neveu-Schwarz vector mesons.

$$\begin{aligned} & \langle 0|V_{-1}^a(k_1, z_1) V_0(k_2, z_2, \epsilon_2) V_{-1}^b(k_3, z_3)c(z_1)c(z_2)c(z_3)|0\rangle \\ & \cdot \langle 0|V_{-1}(k_1, \bar{z}_1, \epsilon_1) V_0(k_2, \bar{z}_2, \epsilon_2) V_{-1}(k_3, \bar{z}_3, \epsilon_3)c(\bar{z}_1)c(\bar{z}_2)c(\bar{z}_3)|0\rangle \\ & = \delta^{ab}\epsilon_2^{\mu\nu}k_{3\mu}[k_{3\nu}\epsilon_1\cdot\epsilon_3 + \epsilon_{3\nu}\epsilon_1\cdot k_2 + \epsilon_{1\nu}\epsilon_3\cdot k_1]. \end{aligned} \quad (5.8)$$

For the polarizations $\epsilon_1^+, \epsilon_3^+$ coupling to the graviton $h_{2\mu\nu}$, the amplitude (5.8) vanishes, as we would expect from angular momentum conservation in the coupling of two spin one mesons (with momenta of opposite sign) with a spin two particle. The proof is as follows: from (3.6) we find the identity

$$\begin{aligned} & (-\epsilon_3^{+\rho}k_3^\sigma + k_3^\rho\epsilon_3^{+\sigma})\eta^{\mu\nu} + (\epsilon_3^{+\rho}k_3^\nu - k_3^\rho\epsilon_3^{+\nu})\eta^{\mu\sigma} + (-\epsilon_3^{+\sigma}k_3^\nu + k_3^\sigma\epsilon_3^{+\nu})\eta^{\mu\rho} \\ & = i(\epsilon_3^{+\mu}k_{3\omega} - k_3^\mu\epsilon_{3\omega}^+)\epsilon^{\omega\rho\sigma\nu} \end{aligned} \quad (5.9)$$

Multiplying (5.9) by $\epsilon_{1\rho}^+k_{1\sigma}$ we find

$$\begin{aligned} & k_1^\mu\epsilon_3^{+\nu}\epsilon_1^+\cdot k_2 + k_3^\mu k_1^\nu\epsilon_1^+\cdot\epsilon_3^+ + k_3^\mu\epsilon_1^\nu\epsilon_3\cdot k_2 \\ & = -k_1^\nu\epsilon_3^{+\mu}\epsilon_1^+\cdot k_2 - k_3^\nu k_1^\mu\epsilon_1^+\cdot\epsilon_3^+ - k_3^\nu\epsilon_1^\mu\epsilon_3\cdot k_2 - \eta^{\mu\nu}\epsilon_1^+\cdot k_3\epsilon_3^+\cdot k_1 \end{aligned} \quad (5.10)$$

From (5.10) and (5.3b) we find

$$h_2^{\mu\nu}k_{3\mu}[k_{3\nu}\epsilon_1^+\cdot\epsilon_3^+ + \epsilon_{3\nu}^+\epsilon_1^+\cdot k_2 + \epsilon_{1\nu}^+\epsilon_3^+\cdot k_1] = 0. \quad (5.11)$$

In the case of vector mesons from the Ramond-Ramond sector, the corresponding amplitude vanishes identically. See (5.14).

For the polarizations $\epsilon_1^+, \epsilon_3^+$ coupling to the dilaton, we use (3.18) and (5.5) to show that the amplitude (5.8) is given by the non-zero expression

$$\begin{aligned} & \delta^{ab} D_2^{\mu\nu} k_{3\mu} [k_{3\nu} \epsilon_1^+ \cdot \epsilon_3^+ + \epsilon_{3\nu}^+ \epsilon_1^+ \cdot k_2 + \epsilon_{1\nu}^+ \epsilon_3^+ \cdot k_1] \\ &= \frac{1}{\sqrt{2}} \epsilon_1^+ \cdot k_3 \epsilon_3^+ \cdot k_1 \end{aligned} \quad (5.12a)$$

$$= \frac{2}{\sqrt{2}} \epsilon_{2\mu}^+ \epsilon_{2\nu}^- k_{3\mu} [\epsilon_{3\nu}^+ \epsilon_1^+ \cdot k_2 + \epsilon_{1\nu}^+ \epsilon_3^+ \cdot k_1]. \quad (5.12b)$$

The equality of (5.12a,b) follows from

$$\begin{aligned} & 2\epsilon_2^+ \cdot k_3 [\epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2 + \epsilon_2^- \cdot \epsilon_1^+ \epsilon_3^+ \cdot k_1] \\ &= 2\epsilon_2^+ \cdot k_3 [ik_{2\mu} \epsilon_{1\sigma}^+ \epsilon_{2\kappa}^- \epsilon_{3\lambda}^+ \epsilon^{\mu\sigma\kappa\lambda}] \\ &= 2\epsilon_{2\rho}^+ \epsilon_{2\kappa}^- k_{2\mu} \epsilon_{1\sigma}^+ \epsilon_{3\lambda}^+ k_3^\rho i\epsilon^{\mu\sigma\kappa\lambda} \\ &= 2[\epsilon_{2\rho}^+ \epsilon_{2\kappa}^- k_{2\mu} \epsilon_{1\sigma}^+ i\epsilon_\lambda^{\rho\alpha\beta} \epsilon_{3\alpha}^+ k_{3\beta} i\epsilon^{\mu\sigma\kappa\lambda} \\ & \quad + \epsilon_{2\rho}^+ \epsilon_{2\kappa}^- k_{2\mu} \epsilon_{1\sigma}^+ \epsilon_3^{\rho\lambda} k_{3\lambda} i\epsilon^{\mu\sigma\kappa\lambda}] \\ &= 2[\epsilon_1^+ \cdot \epsilon_2^+ \epsilon_3^+ \cdot k_2 \epsilon_2^- \cdot k_3 + \epsilon_2^+ \cdot \epsilon_2^- \epsilon_3^+ \cdot k_1 \epsilon_1^+ \cdot k_3 + \epsilon_2^+ \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2 \epsilon_2^- \cdot k_1] \\ &= 2\epsilon_2^+ \cdot \epsilon_2^- \epsilon_3^+ \cdot k_1 \epsilon_1^+ \cdot k_3 \\ &= \epsilon_3^+ \cdot k_1 \epsilon_1^+ \cdot k_3. \end{aligned} \quad (5.12c)$$

Similarly, for the polarizations $\epsilon_1^+, \epsilon_3^+$ coupling to the pseudoscalar described by the antisymmetric tensor, we have

$$\begin{aligned} & \delta^{ab} B_2^{\mu\nu} k_{3\mu} [k_{3\nu} \epsilon_1^+ \cdot \epsilon_3^+ + \epsilon_{3\nu}^+ \epsilon_1^+ \cdot k_2 + \epsilon_{1\nu}^+ \epsilon_3^+ \cdot k_1] \\ &= \frac{1}{4} \epsilon_1^+ \cdot k_3 \epsilon_3^+ \cdot k_1 \end{aligned} \quad (5.13a)$$

$$= \frac{1}{2} \epsilon_{2\mu}^+ \epsilon_{2\nu}^- k_{3\mu} [\epsilon_{3\nu}^+ \epsilon_1^+ \cdot k_2 + \epsilon_{1\nu}^+ \epsilon_3^+ \cdot k_1]. \quad (5.13b)$$

For Ramond-Ramond mesons, the amplitudes corresponding to (5.12,13) vanish:

$$\begin{aligned} & \langle 0 | V_{-\frac{1}{2}}^{(1)}(k_1, z_1) V_{-1}(k_2, z_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\ & \cdot \langle 0 | V_{-\frac{1}{2}}^{(1)}(k_1, \bar{z}_1) V_{-1}(k_2, \bar{z}_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, \bar{z}_3) c(\bar{z}_1) c(\bar{z}_2) c(\bar{z}_3) | 0 \rangle \\ &= u^{2\delta}(k_1) u^{2\gamma}(k_1) \epsilon_{2\mu} \gamma_{\gamma\alpha}^\mu u^{1\alpha}(k_3) u^{1\beta}(k_3) \epsilon_{2\nu} (\gamma^\nu)_{\beta\delta}^T \frac{1}{2} f^k f^{\dot{n}} C_{\dot{n}\ell}^{-1} f^\ell f^m C_{mk}^{-1} \\ &= 0. \end{aligned} \quad (5.14)$$

since only $\gamma_{\gamma\alpha}^\mu$ and $\gamma_{\gamma\dot{\alpha}}^\mu$ are non-zero. From (5.12,13,14) we see the Ramond mesons couple differently to the dilaton (and to the antisymmetric tensor) from the Neveu-Schwarz mesons. Thus in the superstring model of Ref. [13], with $D = 4$, $N = 8$ supergravity that has 28 abelian massless vector mesons, we find 12 of these couple to the dilaton as in (5.12) and the other 16 have zero tree level coupling to the dilaton. This feature already appears in $D = 10$, $N = 2$ supergravity^[14], and is useful in recent non-perturbative treatments of the superstring^[24].

Using the four-dimensional covariant notation developed in sect. 3, we also give the tree amplitude for the *general graviton tensor coupling to two fermions*:

$$\begin{aligned}
& \langle 0 | V_{-\frac{1}{2}}^{(2)}(k_1, z_1) V_{-1}(k_2, z_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\
& \cdot \langle 0 | V_{-1}(k_1, \bar{z}_1, \epsilon_1) V_0(k_2, \bar{z}_2, \epsilon_2) V_{-1}(k_3, \bar{z}_3, \epsilon_3) c(\bar{z}_1) c(\bar{z}_2) c(\bar{z}_3) | 0 \rangle \\
& = u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu} (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) \\
& \quad \cdot \epsilon_{2\nu} [k_3^\nu \epsilon_1 \cdot \epsilon_3 + \epsilon_3^\nu \epsilon_1 \cdot k_2 + \epsilon_1^\nu \epsilon_3 \cdot k_1] f^{\dot{n}} f^\ell C_{\dot{n}\ell}^{-1}.
\end{aligned} \tag{5.15}$$

(5.15) describes many processes. In the following the extended supersymmetry indices \dot{n}, ℓ are suppressed. Using the three-point on-shell kinematics from sect. 5, we find for example, the one graviton – two gravitino string tree amplitudes:

$$\begin{aligned}
& \epsilon_{2\mu}^+ \epsilon_{2\nu}^+ u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu} (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) [k_3^\nu \epsilon_1^- \cdot \epsilon_3^+ + \epsilon_3^{+\nu} \epsilon_1^- \cdot k_2 + \epsilon_1^{-\nu} \epsilon_3^+ \cdot k_1] \\
& = h_{2\mu\nu}^+ \psi_\rho^{-\dot{\gamma}}(k_1) (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} \psi_\lambda^{+\alpha}(k_3) [k_3^\nu \eta^{\rho\lambda} + k_1^\lambda \eta^{\nu\rho}]
\end{aligned} \tag{5.16a}$$

and

$$\begin{aligned}
& \epsilon_{2\mu}^- \epsilon_{2\nu}^- u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu} (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) [k_3^\nu \epsilon_1^- \cdot \epsilon_3^+ + \epsilon_3^{+\nu} \epsilon_1^- \cdot k_2 + \epsilon_1^{-\nu} \epsilon_3^+ \cdot k_1] \\
& = h_{2\mu\nu}^- \psi_\rho^{-\dot{\gamma}}(k_1) (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} \psi_\lambda^{+\alpha}(k_3) [k_3^\nu \eta^{\rho\lambda} + k_2^\rho \eta^{\lambda\nu}].
\end{aligned} \tag{5.16b}$$

Note in (5.16) that the specific non-zero helicity couplings appear automatically. Also from (5.15), we find the one graviton – two spin one-half fermion amplitudes

$$\begin{aligned}
& u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu}^+ (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) \epsilon_{2\nu}^+ [k_3^\nu \epsilon_1^+ \cdot \epsilon_3^- + \epsilon_3^{-\nu} \epsilon_1^+ \cdot k_2 + \epsilon_1^{+\nu} \epsilon_3^- \cdot k_1] \\
& = h_{2\mu\nu}^+ \chi_\rho^{+\dot{\gamma}}(k_1) (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} \chi_\lambda^{-\alpha}(k_3) [k_3^\nu \eta^{\rho\lambda} + k_2^\rho \eta^{\lambda\nu}];
\end{aligned} \tag{5.17a}$$

$$\begin{aligned}
& u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu}^- (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) \epsilon_{2\nu}^- [k_3^\nu \epsilon_1^+ \cdot \epsilon_3^- + \epsilon_3^{-\nu} \epsilon_1^+ \cdot k_2 + \epsilon_1^{+\nu} \epsilon_3^- \cdot k_1] \\
& = h_{2\mu\nu}^- \chi_\rho^{+\dot{\gamma}}(k_1) (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} \chi_\lambda^{-\alpha}(k_3) [k_3^\nu \eta^{\rho\lambda} + k_1^\lambda \eta^{\nu\rho}].
\end{aligned} \tag{5.17b}$$

For spinors with parallel polarizations, the amplitude corresponding to (5.15) vanishes as in (5.14):

$$\begin{aligned}
& \langle 0 | V_{-\frac{1}{2}}^{(1)}(k_1, z_1) V_{-1}(k_2, z_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\
& \cdot \langle 0 | V_{-1}(k_1, \bar{z}_1, \epsilon_1) V_0(k_2, \bar{z}_2, \epsilon_2) V_{-1}(k_3, \bar{z}_3, \epsilon_3) c(\bar{z}_1) c(\bar{z}_2) c(\bar{z}_3) | 0 \rangle \\
& = 0.
\end{aligned} \tag{5.18}$$

(5.15,18) show for example the graviton makes no transition between spin $\frac{3}{2}$ and spin $\frac{1}{2}$, since from (5.15), $u^{2\dot{\gamma}}(k_1) \epsilon_{2\mu}^+ (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) \epsilon_{2\nu}^+ [k_3^\nu \epsilon_1^+ \cdot \epsilon_3^+ + \epsilon_3^{+\nu} \epsilon_1^+ \cdot k_2 + \epsilon_1^{+\nu} \epsilon_3^+ \cdot k_1] = 0$, and from (5.18), $u^{1\dot{\gamma}}(k_1) \epsilon_{2\mu}^+ \gamma_{\dot{\gamma}\alpha}^\mu u^{1\alpha}(k_3) \epsilon_{2\nu}^+ [k_3^\nu \epsilon_1^- \cdot \epsilon_3^+ + \epsilon_3^{+\nu} \epsilon_1^- \cdot k_2 + \epsilon_1^{-\nu} \epsilon_3^+ \cdot k_1] = 0$, etc. Amplitudes for the remaining combinations of spinor polarizations, including spinors for the left instead of right-movers also occur and are similar to (5.15,17). Additional spin $\frac{1}{2}$ fermions also appear from left-moving scalars and right-moving spinors, and visa versa.

6. Non-abelian coupling

In this section, we discuss a set of tree amplitudes involving massless gauge boson states coming from the Ramond-Ramond sector, which exhibit a non-abelian structure. In the spirit of previous investigations [15], the gauge symmetry is enlarged by modifying the choice of the worldsheet supercurrent. Although such states are forbidden by the conventional analysis of [16], nonetheless we find their kinematic structure interesting, and suggest one possible mechanism by which they could be incorporated into an interacting string model satisfying space-time unitarity. This mechanism involves a non-hermitian piece of the internal supercurrent $\tilde{F}(z)$. The supercurrent is given by

$$F(z) = a_\mu(z)\psi^\mu(z) + \bar{F}(z) \quad (6.1)$$

where $0 \leq \mu \leq 3$ and we choose

$$\bar{F}(z) = \tilde{\gamma}^5 \otimes \hat{F}(z) + \tilde{\gamma}^5 \otimes \tilde{\Gamma}^7 \otimes \tilde{F}(z) \quad (6.2)$$

where $\hat{F}(z) \equiv (-\frac{i}{6}) \frac{1}{\sqrt{\frac{c_\psi}{2}}} f_{abc} \psi^a(z) \psi^b(z) \psi^c(z)$ and f_{abc} given by the totally antisymmetric structure constants of $SU(2) \otimes SU(2)$ with $f_{abc} f_{abe} = c_\psi \delta_{ce}$. For $\mathcal{S}(z) \equiv f^\ell \Sigma_\ell(z)$ and $\tilde{\mathcal{S}}(z) \equiv \frac{1}{2} e^{-\frac{i\pi}{4}} f^\ell \Sigma_\ell(z)$, we have

$$\hat{F}(z) \mathcal{S}(\zeta) = (z - \zeta)^{-\frac{3}{2}} \tilde{\mathcal{S}}(\zeta) + \dots \quad (6.3)$$

$$\hat{F}(z) \tilde{\mathcal{S}}(\zeta) = (z - \zeta)^{-\frac{3}{2}} \frac{1}{4} \mathcal{S}(\zeta) + \dots \quad (6.4)$$

where we use operator products of the form

$$f_{abc} \psi^a(z) \psi^b(z) \psi^c(z) \Sigma_\ell(\zeta) = (z - \zeta)^{-\frac{3}{2}} f_{abc} \frac{1}{2\sqrt{2}} (\Gamma^a \Gamma^b \Gamma^c)_\ell^{\dot{\ell}} \Sigma_{\dot{\ell}}(\zeta) + \dots \quad (6.5)$$

$\tilde{F}(z)$ corresponds to the remaining internal degrees of freedom with $c = 6$. We assume there exists an anti-periodic supercurrent, and a pair of states in the Ramond sector of this piece of the internal system corresponding to weight zero conformal spin fields $V(z)$ and $U(z)$ such that

$$\begin{aligned} \tilde{F}(z) V(\zeta) &= (z - \zeta)^{-\frac{3}{2}} U(\zeta) + \dots \\ \tilde{F}(z) U(\zeta) &= (z - \zeta)^{-\frac{3}{2}} (-\frac{c}{24}) V(\zeta) + \dots \end{aligned} \quad (6.5)$$

(6.5) corresponds to a non-hermitian choice for the operator \tilde{F}_0 , that is to say non-hermitian with respect to a vector space of states with positive-definite inner product. This is because in such a space, the eigenvalues of the square of a hermitian operator are non-negative, but from (6.5) we see $\tilde{F}_0^2 V(0)|0\rangle = -\frac{1}{4} V(0)|0\rangle$, i.e., a negative eigenvalue. Note that the definition of the hermitian adjoint for any operator A , which is

$$(A\psi_a, \psi_b) = (\psi_a, A^\dagger \psi_b), \quad (6.6a)$$

depends on the definition of the inner product

$$(\psi_a, \psi_b), \quad (6.6b)$$

since (6.6b) is used to evaluate (6.6a). One example of a conformal field theory where F_0^2 can take on negative eigenvalues is the b, c, β, γ superconformal ghost system; here F_0 is not hermitian with respect to a non-vanishing inner product, given for example by the adjoint state defined as $(c_1|0)^\dagger = \langle 0|c_{-1}c_0$. (Although F_0 is hermitian with respect to a vanishing norm, given for example by the adjoint state defined as $(c_1|0)^\dagger = \langle 0|c_{-1}$.) Another example is the combined system $(T_m, (b_1, c_1))$ with $N = 1$ superconformal symmetry with $c = 15$ of Berkovits and Vafa^[17] representing the matter system of an $N = 1$ fermionic string whose states physical states are in one-to-one correspondence with the states of the bosonic string with $c = 26$.

We also remark that since the norm of a state $\tilde{F}_0|\Psi\rangle$ is given by $||\tilde{F}_0|\Psi\rangle|| = \langle \Psi|\tilde{F}_0^\dagger\tilde{F}_0|\Psi\rangle$, that therefore non-hermitian \tilde{F}_0 does not imply $\tilde{F}_0|\Psi\rangle$ is a negative norm state, even though $\langle \Psi|\tilde{F}_0^2|\Psi\rangle < 0$.

The states of the Ramond sector are created by conformal fields called spin fields^[18–22]. The spin fields are nonlocal with respect to the worldsheet supercurrent T_F because they make states in the Ramond sector of the theory, whereas superconformal fields corresponding to states in the Neveu-Schwarz sector are local with respect to T_F . In general, the operator product of a spin field with any fermionic part of a superfield is nonlocal (i.e. double-valued) since spin fields change the boundary condition on fermion fields between periodic and anti-periodic.

In the Ramond sector, the operator products of the spin fields with T_F , have a $(z-\zeta)^{-\frac{3}{2}}$ singularity; all except for the “ground states”, (i.e. those for which $h = \frac{c}{24}$), which have only a $(z-\zeta)^{-\frac{1}{2}}$ singularity with T_F . Since L_0 and F_0 commute, all excited states are in pairs $S^\pm(z)|0\rangle$ related by F_0 , only the “ground states” need not be paired. We have

$$\begin{aligned} |h^+\rangle &= S^+(0)|0\rangle \\ |h^-\rangle &= S^-(0)|0\rangle = F_0|h^+\rangle \\ (h - \frac{c}{24})|h^+\rangle &= F_0|h^-\rangle \\ L_0|h^\pm\rangle &= h|h^\pm\rangle \end{aligned}$$

or in terms of the fields:

$$\begin{aligned} F(z)S^+(\zeta) &= (z-\zeta)^{-\frac{3}{2}}S^-(\zeta) \\ F(z)S^-(\zeta) &= [h - \frac{c}{24}](z-\zeta)^{-\frac{3}{2}}S^+(\zeta). \end{aligned}$$

If $h = \frac{c}{24}$, global worldsheet supersymmetry is unbroken in the Ramond sector; where F_0 is the global supersymmetry charge satisfying the global supersymmetry algebra $F_0^2 = L_0 - \frac{c}{24}$. In this case the states need not come in pairs, i.e. only $|h^-\rangle$ survives since if F_0 is hermitian, then for $h = \frac{c}{24}$ we have $\langle h^-|h^- \rangle = \langle h^+|F_0^2|h^+ \rangle = \langle h^+|L_0 - \frac{c}{24}|h^+ \rangle = 0$, i.e. $|h^-\rangle$ is null, i.e. has zero norm.

The operator product expansions for the pair of spin fields $U(z), V(z)$ are chosen to be

$$\begin{aligned}
V(z)V(\zeta) &= (z-\zeta)^0 C_0(\zeta) + \dots \\
U(z)U(\zeta) &= (z-\zeta)^0 C'_0(\zeta) + \dots \\
V(z)U(\zeta) &= \dots + (z-\zeta)^{-\frac{1}{2}} C_{-\frac{1}{2}}(\zeta) + \dots \\
U(z)V(\zeta) &= \dots - (z-\zeta)^{-\frac{1}{2}} C_{-\frac{1}{2}}(\zeta) + \dots
\end{aligned} \tag{6.7}$$

where the subleading terms are less singular by integer powers of $(z-\zeta)$, and for example $C_{-\frac{1}{2}}(z)$ is some dimension $-\frac{1}{2}$ operator whose coefficient may be zero apriori, and which is understood to represent $\tilde{\gamma}^5 \otimes \tilde{\Gamma}^7 \otimes C_{-\frac{1}{2}}(z)$; and $C_0(z)$ has weight zero, etc. (6.7) together with (4.2) are consistent with locality of the string vertices given in (6.1).

For $\Sigma^{(1)}(z) \equiv \mathcal{S}(z)V(z)$ and $\tilde{\Sigma}^{(1)}(z) \equiv \tilde{\mathcal{S}}(z)V(z) + \mathcal{S}(z)U(z)$, BRST invariant vertex operators for the Ramond states in the canonical $q = -\frac{1}{2}$ superconformal ghost picture are given for u, v in (3.2),(3.3) by

$$V_{-\frac{1}{2}}^{(1)}(k, \zeta) = [u^{1\alpha}(k)S_\alpha(\zeta)\Sigma^{(1)}(\zeta) - v^{1\dot{\alpha}}(k)S_{\dot{\alpha}}(\zeta)\tilde{\Sigma}^{(1)}(\zeta)] e^{ik \cdot X(\zeta)} e^{-\frac{1}{2}\phi(\zeta)}, \tag{6.8a}$$

and for $\Sigma^{(2)}(z) \equiv \tilde{\mathcal{S}}(z)V(z)$ and $\tilde{\Sigma}^{(2)}(z) \equiv \frac{1}{4}\mathcal{S}(z)V(z) - \tilde{\mathcal{S}}(z)U(z)$,

$$V_{-\frac{1}{2}}^{(2)}(k, \zeta) = [u^{2\dot{\alpha}}(k)S_{\dot{\alpha}}(\zeta)\Sigma^{(2)}(\zeta) - v^{2\alpha}(k)S_\alpha(\zeta)\tilde{\Sigma}^{(2)}(\zeta)] e^{ik \cdot X(\zeta)} e^{-\frac{1}{2}\phi(\zeta)}. \tag{6.8b}$$

We note that within the fermion scattering amplitudes, the modified fermion emission vertex given above effectively normalizes the Ramond-Ramond bosons to 1 not k^0 , i.e. the same normalization as the conventional gauge bosons found in the NS-NS sector, a required feature when both kinds of bosons are used together to form the adjoint representation of a bigger group. Consider the following amplitude. On the left-moving side we have

$$\begin{aligned}
&\langle 0 | V_{-\frac{1}{2}}^{(1)}(k_1, z_1) V_{-1}(k_2, z_2, \epsilon_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\
&= \langle 0 | [u^{1\gamma}(k_1)S_\gamma(z_1)\Sigma^{(1)}(z_1) - v^{1\dot{\gamma}}(k_1)S_{\dot{\gamma}}(z_1)\tilde{\Sigma}^{(1)}(z_1)] e^{ik_1 \cdot X(z_1)} e^{-\frac{1}{2}\phi(z_1)} \\
&\quad \cdot [\epsilon_{2\mu}\psi^\mu(z_2) e^{ik_2 \cdot X(z_2)}] e^{-\phi(z_2)} \\
&\quad \cdot [u^{1\alpha}(k_3)S_\alpha(z_3)\Sigma^{(1)}(z_3) - v^{1\dot{\alpha}}(k_3)S_{\dot{\alpha}}(z_3)\tilde{\Sigma}^{(1)}(z_3)] e^{ik_3 \cdot X(z_3)} e^{-\frac{1}{2}\phi(z_3)} | 0 \rangle \\
&\quad \cdot \langle 0 | c(z_1) c(z_2) c(z_3) | 0 \rangle \\
&= -[u^{1\gamma}(k_1)\epsilon_{2\mu}\gamma_{\gamma\dot{\alpha}}^\mu v^{1\dot{\alpha}}(k_3) + v^{1\dot{\gamma}}(k_1)\epsilon_{2\mu}\gamma_{\dot{\gamma}\alpha}^\mu u^{1\alpha}(k_3)] \frac{1}{\sqrt{2}} \\
&\quad \cdot f^n f^\ell C_{n\ell}^{-1} \frac{1}{2} e^{-\frac{i\pi}{4}}
\end{aligned} \tag{6.9}$$

On the right-moving side we consider

$$\begin{aligned}
& \langle 0 | V_{-\frac{1}{2}}^{(1)}(k_1, z_1) V_{-1}^a(k_2, z_2) V_{-\frac{1}{2}}^{(1)}(k_3, z_3) c(z_1) c(z_2) c(z_3) | 0 \rangle \\
&= \langle 0 | [u^{1\delta}(k_1) S_\delta(z_1) \Sigma^{(1)}(z_1) \\
&\quad - v^{1\dot{\delta}}(k_1) S_{\dot{\delta}}(z_1) \tilde{\Sigma}^{(1)}(z_1)] e^{ik_1 \cdot X(z_1)} e^{-\frac{1}{2}\phi(z_1)} \\
&\quad \cdot [\tilde{\gamma}^5 \otimes \psi^a(z_2) e^{ik_2 \cdot X(z_2)}] e^{-\phi(z_2)} \\
&\quad \cdot [u^{1\beta}(k_3) S_\beta(z_3) \Sigma^{(1)}(z_3) \\
&\quad - v^{1\dot{\beta}}(k_3) S_{\dot{\beta}}(z_3) \tilde{\Sigma}^{(1)}(z_3)] e^{ik_3 \cdot X(z_3)} e^{-\frac{1}{2}\phi(z_3)} | 0 \rangle \\
&\quad \cdot \langle 0 | c(z_1) c(z_2) c(z_3) | 0 \rangle \\
&= u^{1\delta}(k_1) C_{\delta\beta}^{-1} u^{1\beta}(k_3) f^k f^m (-\alpha_{km}^a) \frac{1}{\sqrt{2}}
\end{aligned} \tag{6.10}$$

where in (6.10) we have assumed $C'_0 = \frac{i}{4}C_0$. Combining the left and right-moving pieces (6.9) and (6.10) we find up to an overall normalization constant

$$\begin{aligned}
A_3 &= i f^k f^n C_{n\dot{\ell}}^{-1} f^{\dot{\ell}} f^m \alpha_{mk}^a \frac{1}{2} \\
&\quad \cdot \epsilon_{2\mu}^- [u^{1\delta}(k_1) u^{1\gamma}(k_1) (-i\sigma^2 \bar{\sigma}^\mu)_{\gamma\dot{\alpha}} v^{1\dot{\alpha}}(k_3) u^{1\beta}(k_3) (i\sigma^2)_{\beta\delta} \frac{1}{\sqrt{2}} \\
&\quad + u^{1\delta}(k_1) v^{1\dot{\gamma}}(k_1) (-i\sigma^2 \sigma^\mu)_{\dot{\gamma}\alpha} u^{1\alpha}(k_3) u^{1\beta}(k_3) (i\sigma^2)_{\beta\delta} \frac{1}{\sqrt{2}}] \\
&= i f^k f^n C_{n\dot{\ell}}^{-1} f^{\dot{\ell}} f^m \alpha_{mk}^a \frac{1}{2} \\
&\quad \cdot \text{tr} [(-\epsilon_{1\rho}^+ k_{1\sigma} \bar{\sigma}^\rho \sigma^\sigma \sigma^2) (-i\sigma^2 \bar{\sigma}^\mu) (\epsilon_{3\lambda}^+ \sigma^\lambda \sigma^2) (i\sigma^2) \\
&\quad + (\epsilon_{1\rho}^+ \bar{\sigma}^\rho \sigma^2) (-i\sigma^2 \sigma^\mu) (-\epsilon_{3\lambda}^+ k_{3\kappa} \bar{\sigma}^\lambda \sigma^\kappa \sigma^2) (i\sigma^2)] \\
&= i f_{IaJ} [\epsilon_1^+ \cdot \epsilon_2^- \epsilon_3^+ \cdot k_1 + \epsilon_2^- \cdot \epsilon_3^+ \epsilon_1^+ \cdot k_2]
\end{aligned} \tag{6.11}$$

which is the three-gluon tree coupling for this set of polarizations, as $\epsilon_1^+ \cdot \epsilon_3^+ \epsilon_2^- \cdot k_3 = 0$. The structure constants $f_{IaJ} = 2 \text{tr}(M_I^\dagger M_J \alpha^a)$ form the (2,2,2,2) representation^[6] of $SU(2)^4$ and suggest that the symmetry group is enhanced to $SO(8)$ from its symmetric subgroup $SU(2)^4$. Generalizations to other polarizations must be constructed consistent with the required couplings^[23]. Also, couplings to other states such as the graviton must be of the conventional form. This can involve modifying the superconformal fields corresponding to the Neveu-Schwarz states by the internal \tilde{F} conformal theory as well as the above modification of the spin fields. For example, in (6.9) and (6.10) we can replace $V_{-1}(k_2, z_2, \epsilon_2) = \epsilon \cdot \psi e^{ik \cdot X}$ and $V_{-1}^a(k_2, z_2) = \tilde{\gamma} \otimes \psi^a e^{ik \cdot X}$ by forms similar to $\epsilon \cdot \psi C_0 + (\epsilon \cdot a + k \cdot \psi \epsilon \cdot \psi) C_{-\frac{1}{2}}$ and $\tilde{\gamma} \otimes \psi^a C_0 + (k \cdot \psi \psi^a - \frac{i}{2} f_{abc} \psi^b \psi^c) C_{-\frac{1}{2}}$ respectively. Here $C_{-\frac{1}{2}}$ and C_0 form the lower and upper components of a superconformal field, and the modified vertices remain BRST invariant. In this way, the states in the hermitian part of the theory could be in one-to-one correspondence with the physical states in the entire theory. These modified vertices yield expressions in (6.11) with all polarizations present.

7. Modular invariance

To describe consistent interacting strings, one must in general check that all the scattering amplitudes are modular invariant, finite, and unitary. A guide to this program is the calculation of the partition function, i.e. the one-loop cosmological constant Λ , which can be checked for modular invariance, albeit a quantity equal to zero. For closed strings, the one-loop cosmological constant is defined by

$$\Lambda \equiv \frac{1}{2} \text{tr} \ln \Delta^{-1} \quad (7.1)$$

where

$$\Delta^{-1} = \alpha'(p^2 + m^2); \quad \frac{1}{2}\alpha'm^2 = \alpha'm_L^2 + \alpha'm_R^2 \quad (7.2)$$

so in D space-time dimensions, for $\omega = e^{2\pi i\tau}$,

$$\begin{aligned} \Lambda = & -\frac{1}{2}(2\pi)^{-1}(\alpha')^{-\frac{D}{2}} \int_F d^2\tau (Im\tau)^{-2-\frac{1}{2}(D-2)} \\ & \cdot \sum_{\text{all sectors}} \text{tr}[\bar{\omega}^{\alpha'm_L^2} \omega^{\alpha'm_R^2} \text{possible projections}]. \end{aligned} \quad (7.3)$$

F is a fundamental region of the modular group: $\frac{1}{2} \leq Re\tau \leq \frac{1}{2}; |\tau| > 1$. The unitary $D = 4$, $N = 8$ superstring model considered in [13], which incorporates (4.2b) as *part* of the internal field theory, has its one-loop modular invariant partition function given by

$$\begin{aligned} \Lambda = & -(4\pi\alpha'^2)^{-1} \int_F d^2\tau (Im\tau)^{-3} |f(\omega)|^{-12} |\omega|^{-\frac{1}{2}} \\ & \cdot \frac{1}{4} [\theta_3^4 - \theta_4^4 - \theta_2^4] [\bar{\theta}_3^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4] \\ & \cdot \left[\frac{1}{2} (|\theta_3|^{12} + |\theta_4|^{12} + |\theta_2|^{12}) |f(\omega)|^{-12} |\omega|^{-\frac{1}{2}} \right]. \end{aligned} \quad (7.4)$$

For the $D = 4$, $N = 8$ model considered in sect. 6, the partition function computed for the degrees of freedom denoted in (4.2) by a_μ, ψ^μ and ψ^a with $\alpha'm_L^2 = L_0 - \frac{c}{24}$, etc. is

$$\begin{aligned} \Lambda = & -(4\pi\alpha'^2)^{-1} \int_F d^2\tau (Im\tau)^{-3} |f(\omega)|^{-12} |\omega|^{-\frac{1}{2}} \\ & \cdot \frac{1}{4} [\theta_3^4 - \theta_4^4 - \theta_2^4] [\bar{\theta}_3^4 - \bar{\theta}_4^4 - \bar{\theta}_2^4]. \end{aligned} \quad (7.5)$$

We see (7.5) is already modular invariant using the transformation properties under $SL(2, I)$ given by

$$\begin{aligned} \tau \rightarrow \tau + 1 : & \quad \theta_3 \rightarrow \theta_4; \theta_4 \rightarrow \theta_3; \theta_2 \rightarrow e^{i\frac{\pi}{4}}\theta_2; Im\tau \rightarrow Im\tau \\ \tau \rightarrow -\frac{1}{\tau} : & \quad \theta_2 \rightarrow (-i\tau)^{\frac{1}{2}}\theta_4; \theta_4 \rightarrow (-i\tau)^{\frac{1}{2}}\theta_2; \theta_3 \rightarrow (-i\tau)^{\frac{1}{2}}\theta_3; Im\tau \rightarrow |\tau|^{-2}Im\tau; \\ & \quad \omega^{\frac{1}{24}}f(\omega) \rightarrow (-i\tau)^{\frac{1}{2}}\omega^{\frac{1}{24}}f(\omega) \end{aligned} \quad (7.6)$$

where $f(\omega) = \prod_{n=1}^{\infty} (1 - \omega^n)$ is related to the Dedekind eta function $\eta(\omega) = \omega^{\frac{1}{24}}f(\omega)$, $\omega = e^{2\pi i\tau}$ and $\theta_i(0|\tau)$ are the Jacobi theta functions. We suggest that the inclusion of the remaining internal conformal field theory will leave the modular invariance of (7.5) unchanged in the theory postulated in sect.6.; and the physical states may be in one to one correspondence with the states of the partition function given in (7.5). This latter feature of adding conformal fields to the matter sector without changing the spectrum has been discussed in a different context in [17].

8. Conclusions

We have developed a formalism for studying four-dimensional massless spin fields, and computed their BRST invariant tree amplitudes. In sect.'s 6 and 7, the physical states in the $q = -\frac{1}{2}$ ghost picture, given by say (6.8a),

$$V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle \quad (8.1)$$

satisfy, for F_0 given in (6.2):

$$F_0 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0, \quad (8.2)$$

which is the physical state condition in the old covariant formalism. We can either check (8.2) explicitly, or recall that it follows from the BRST invariance of (8.1). These states have the additional feature that

$$F_0^{s.t.} V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = -u^{1\alpha}(k) S_\alpha(0) \tilde{\Sigma}(0) e^{ik \cdot X(0)} |0\rangle \neq 0 \quad (8.3a)$$

$$\bar{F}_0 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = u^{1\alpha}(k) S_\alpha(0) \tilde{\Sigma}(0) e^{ik \cdot X(0)} |0\rangle \neq 0, \quad (8.3b)$$

where $F^{s.t.}(z) \equiv a_\mu(z) \psi^\mu(z)$, even though the sum $F_0 = F_0^{s.t.} + \bar{F}_0$ does annihilate the states as is shown in (8.2). Of course (8.3a) still describes massless states since although $k \cdot \gamma v = u \neq 0$, we have

$$-m^2 v = k^2 v = k \cdot \gamma u = 0, \quad (8.4)$$

so that

$$(F_0^{s.t.})^2 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0. \quad (8.5)$$

Now, $F_0 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0$ implies $F_0^2 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0$. Using $\{F_0^{s.t.}, \bar{F}_0\} = 0$ and (8.5), we find

$$F_0^2 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = ((F_0^{s.t.})^2 + \bar{F}_0^2) V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0, \quad (8.6)$$

and thus

$$\bar{F}_0^2 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0. \quad (8.7)$$

So we have $\bar{F}_0^2 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle = 0$, but from (8.3b) that $\bar{F}_0 V_{-\frac{1}{2}}^{(1)}(k, 0)|0\rangle \neq 0$. This is consistent with the observation that \tilde{F}_0 which forms part of \bar{F}_0 is not hermitian. (8.3b) indicates that global worldsheet supersymmetry generated by the charge \bar{F}_0 is broken in the Ramond sector. Consistency of the one-loop amplitudes is being investigated^[23].

The mechanism we have suggested yields a string model with the gauge group derived by enhancing the symmetric subgroup $SU(2)^4$ to $SO(8)$, which although providing non-abelian Ramond-Ramond vector mesons, does not contain a realistic theory. We propose a similar mechanism can be used to generate the gauge symmetry $SU(3) \otimes SU(2) \otimes U(1)$ from a symmetric subgroup such as $SU(2)^2 \otimes U(1)^2$.

Appendix A

SIGMA MATRICES

For the sigma matrices described in (3.1), we have the following symmetry properties

$$\begin{aligned}\sigma^\mu \bar{\sigma}^\nu &= -\sigma^\nu \bar{\sigma}^\mu + 2\eta^{\mu\nu} & \bar{\sigma}^\mu \sigma^\nu &= -\bar{\sigma}^\nu \sigma^\mu + 2\eta^{\mu\nu} \\ \sigma^\mu \bar{\sigma}^\nu &= \eta^{\mu\nu} - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_\rho \bar{\sigma}_\sigma & \bar{\sigma}^\mu \sigma^\nu &= \eta^{\mu\nu} + \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\rho \sigma_\sigma\end{aligned}\quad (A.1)$$

$$\begin{aligned}\sigma^\mu \bar{\sigma}^\nu \sigma^\rho + \sigma^\rho \bar{\sigma}^\nu \sigma^\mu &= 2(\eta^{\mu\nu}\sigma^\rho - \eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu) \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho + \bar{\sigma}^\rho \sigma^\nu \bar{\sigma}^\mu &= 2(\eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu)\end{aligned}\quad (A.2)$$

$$\begin{aligned}\sigma^\mu \bar{\sigma}^\nu \sigma^\rho - \sigma^\rho \bar{\sigma}^\nu \sigma^\mu &= 2i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho - \bar{\sigma}^\rho \sigma^\nu \bar{\sigma}^\mu &= -2i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma\end{aligned}\quad (A.3)$$

so that

$$\begin{aligned}\sigma^\mu \bar{\sigma}^\nu \sigma^\rho &= \eta^{\mu\nu}\sigma^\rho - \eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho &= \eta^{\mu\nu}\bar{\sigma}^\rho - \eta^{\mu\rho}\bar{\sigma}^\nu + \eta^{\nu\rho}\bar{\sigma}^\mu - i\epsilon^{\mu\nu\rho\sigma}\bar{\sigma}_\sigma\end{aligned}\quad (A.4)$$

and the trace properties $\text{tr}\sigma^\mu \bar{\sigma}^\nu = 2\eta^{\mu\nu}$,

$$\begin{aligned}\text{tr}\sigma^\rho \bar{\sigma}^\mu \sigma^\lambda \bar{\sigma}^\kappa &= 2\eta^{\rho\mu}\eta^{\lambda\kappa} - 2\eta^{\rho\lambda}\eta^{\mu\kappa} + 2\eta^{\rho\kappa}\eta^{\mu\lambda} + 2i\epsilon^{\rho\mu\lambda\kappa} \\ \text{tr}\bar{\sigma}^\rho \sigma^\mu \bar{\sigma}^\lambda \sigma^\kappa &= 2\eta^{\rho\mu}\eta^{\lambda\kappa} - 2\eta^{\rho\lambda}\eta^{\mu\kappa} + 2\eta^{\rho\kappa}\eta^{\mu\lambda} - 2i\epsilon^{\rho\mu\lambda\kappa},\end{aligned}\quad (A.5)$$

and

$$\begin{aligned}\text{tr}\sigma^\rho \bar{\sigma}^\sigma \sigma^\mu \bar{\sigma}^\lambda \sigma^\kappa \bar{\sigma}^\nu &= 2[\eta^{\rho\sigma}(\eta^{\mu\lambda}\eta^{\kappa\nu} - \eta^{\mu\kappa}\eta^{\lambda\nu} + \eta^{\mu\nu}\eta^{\lambda\kappa}) \\ &\quad - \eta^{\rho\mu}(\eta^{\sigma\lambda}\eta^{\kappa\nu} - \eta^{\sigma\kappa}\eta^{\lambda\nu} + \eta^{\sigma\nu}\eta^{\lambda\kappa}) \\ &\quad + \eta^{\rho\lambda}(\eta^{\sigma\mu}\eta^{\kappa\nu} - \eta^{\sigma\kappa}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\kappa}) \\ &\quad - \eta^{\rho\kappa}(\eta^{\sigma\mu}\eta^{\lambda\nu} - \eta^{\sigma\lambda}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\lambda}) \\ &\quad + \eta^{\rho\nu}(\eta^{\sigma\mu}\eta^{\lambda\kappa} - \eta^{\sigma\lambda}\eta^{\mu\kappa} + \eta^{\sigma\kappa}\eta^{\mu\lambda}) \\ &\quad + i(\eta^{\rho\sigma}\epsilon^{\mu\lambda\kappa\nu} - \eta^{\sigma\mu}\epsilon^{\lambda\kappa\nu\rho} - \eta^{\lambda\nu}\epsilon^{\rho\sigma\mu\kappa} \\ &\quad - \eta^{\rho\mu}\epsilon^{\sigma\lambda\kappa\nu} + \eta^{\lambda\kappa}\epsilon^{\rho\sigma\mu\nu} + \eta^{\kappa\nu}\epsilon^{\rho\sigma\mu\lambda})] \\ \text{tr}\bar{\sigma}^\rho \sigma^\sigma \bar{\sigma}^\mu \sigma^\lambda \bar{\sigma}^\kappa \sigma^\nu &= 2[\eta^{\rho\sigma}(\eta^{\mu\lambda}\eta^{\kappa\nu} - \eta^{\mu\kappa}\eta^{\lambda\nu} + \eta^{\mu\nu}\eta^{\lambda\kappa}) \\ &\quad - \eta^{\rho\mu}(\eta^{\sigma\lambda}\eta^{\kappa\nu} - \eta^{\sigma\kappa}\eta^{\lambda\nu} + \eta^{\sigma\nu}\eta^{\lambda\kappa}) \\ &\quad + \eta^{\rho\lambda}(\eta^{\sigma\mu}\eta^{\kappa\nu} - \eta^{\sigma\kappa}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\kappa}) \\ &\quad - \eta^{\rho\kappa}(\eta^{\sigma\mu}\eta^{\lambda\nu} - \eta^{\sigma\lambda}\eta^{\mu\nu} + \eta^{\sigma\nu}\eta^{\mu\lambda}) \\ &\quad + \eta^{\rho\nu}(\eta^{\sigma\mu}\eta^{\lambda\kappa} - \eta^{\sigma\lambda}\eta^{\mu\kappa} + \eta^{\sigma\kappa}\eta^{\mu\lambda}) \\ &\quad - i(\eta^{\rho\sigma}\epsilon^{\mu\lambda\kappa\nu} - \eta^{\sigma\mu}\epsilon^{\lambda\kappa\nu\rho} - \eta^{\lambda\nu}\epsilon^{\rho\sigma\mu\kappa} \\ &\quad - \eta^{\rho\mu}\epsilon^{\sigma\lambda\kappa\nu} + \eta^{\lambda\kappa}\epsilon^{\rho\sigma\mu\nu} + \eta^{\kappa\nu}\epsilon^{\rho\sigma\mu\lambda})].\end{aligned}\quad (A.5)$$

Appendix B

GAMMA MATRICES

Higher-dimensional gamma matrices occur naturally in critical superstring theory. In (3.1) and (4.3) we have given a Weyl representation for the four and six-dimensional cases, respectively. In this appendix, we consider in ten dimensions the Γ matrices satisfying $\{\Gamma^M, \Gamma^N\} = 2\eta^{MN}$; $\eta^{MN} = \text{diag}\{-1, 1, \dots, 1\}$. In a Weyl representation, they have only off-diagonal components and can be written as

$$\Gamma^\mu = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \otimes \gamma^\mu; \quad \Gamma^{a+3} = \begin{pmatrix} 0 & \alpha^a \\ -\alpha^a & 0 \end{pmatrix} \otimes I_4; \quad \Gamma^{a+6} = \begin{pmatrix} 0 & \beta^a \\ \beta^a & 0 \end{pmatrix} \otimes \bar{\gamma}^5. \quad (B.1)$$

Here $0 \leq M, N \leq 9$; $0 \leq \mu \leq 3$ and $1 \leq a \leq 3$; and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The six matrices α^a , β^a are antisymmetric and real and satisfy the following algebra:

$$\begin{aligned} \{\alpha^a, \alpha^b\} &= \{\beta^a, \beta^b\} = -2\delta^{ab} \\ [\alpha^a, \beta^b] &= 0, \quad [\alpha^a, \alpha^b] = -2\epsilon_{abc}\alpha^c, \quad [\beta^a, \beta^b] = 2\epsilon_{abc}\beta^c. \end{aligned} \quad (B.2)$$

An explicit representation of these matrices is given in terms of the Pauli matrices:

$$\begin{aligned} \beta^1 &= \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \\ \alpha^1 &= \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}. \end{aligned} \quad (B.3)$$

The hermitian chirality operator is $\Gamma^{11} \equiv \Gamma^0 \dots \Gamma^9 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix} \otimes I_4$, $(\Gamma^{11})^2 = 1$ and $(\bar{\gamma}^5)^2 = -1$, so $\bar{\gamma}^5$ is antihermitian and is given by $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\bar{\gamma}^5$. The matrix direct product is $A \otimes B \equiv \begin{pmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \dots & \dots & \dots \end{pmatrix}$; it follows that $(A \otimes B)(C \otimes D) = AC \otimes BD$.

We denote the index structure of the Γ matrices as $\Gamma^{\mu\mathcal{A}}_{\mathcal{B}}$ and tensors which raise and lower spinor indices are the antisymmetric tensors $C_{\mathcal{AB}}^{-1}$, $C^{\mathcal{AB}}$, the charge conjugation matrices for $SO(9, 1)$. As before, $\chi_{\mathcal{A}} = C_{\mathcal{AB}}^{-1}\chi^{\mathcal{B}}$, $\chi^{\mathcal{A}} = C^{\mathcal{AB}}\chi_{\mathcal{B}}$, $C_{\mathcal{AB}}^{-1}C^{\mathcal{BC}} = \delta_{\mathcal{A}}^{\mathcal{C}}$, $C^{\mathcal{AB}} = -C^{\mathcal{BA}}$, $C_{\mathcal{AB}}^{-1} = -C_{\mathcal{BA}}^{-1}$. Then $A_{\mathcal{A}}B^{\mathcal{A}} = -A^{\mathcal{A}}B_{\mathcal{A}}$. It follows from the definition of the charge conjugation matrix $C_{\mathcal{AB}}^{-1}(\Gamma^\mu)^{\mathcal{B}}_{\mathcal{C}}C^{\mathcal{ED}} = -(\Gamma^{\mu T})_{\mathcal{A}}^{\mathcal{D}}$ that $(\Gamma^\mu)^{\mathcal{D}}_{\mathcal{A}} = (\Gamma^\mu)_{\mathcal{A}}^{\mathcal{D}}$; and that $\Gamma^{\mu\mathcal{AB}}$ and $\Gamma_{\mathcal{AB}}^\mu$ are symmetric in the spinor indices. The exact form of C depends on the representation of the Γ matrices used. From (B.1), we have

$$C = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \otimes C_4 \quad ; \quad C^{-1} = \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} \otimes C_4^{-1} \quad (B.4)$$

where C_4 is the charge conjugation matrix for the four-dimensional γ matrices: $C_4^{-1}(\gamma^\mu)C_4 = -(\gamma^{\mu T})$ and thus $\gamma^{5*} = C^{-1}\gamma^5C$. We will denote $1 \leq \mathcal{A} \leq 32$ as $1 \leq A, \bar{A} \leq 16$.

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